ON THE THEORY OF DIFFRACTION OF CYLINDRICAL ELASTIC WAVES AND WEAK SHOCK WAVES

(K TEORII DIFRAKTSII TSILINDRICHESKIKH UPRUGIKH I SLABYKH UDARNYKH VOLN)

PMM Vol.28, № 2, 1964, pp.296-304

I. G. FILIPPOV
(Moscow)

(Received June 25, 1963)

The unsteady problem of diffraction of weak plane shock waves around a contour of arbitrary shape was investigated and solved in [1 and 2]. In the present paper the results of [1 and 2] are generalized to two-dimensional unsteady problems of diffraction of cylindrical weak shock waves and elastic waves. General theorems are adduced which make possible the investigation and solution of the diffraction problems in the general formulation.

The solutions of a number of diffraction problems are given in terms of quadratures by a common method which includes the second approximation. Some of these problems were solved by other authors in the linear approximation by various artificial means.

1. Formulation and solution of two-dimensional problems. We consider the diffraction of a cylindrical weak shock wave at an infinite cylinder of arbitrary cross section when the wave front is parallel to the axis of the cylinder. The diffraction problem is clearly two-dimensional.

The intensity of the incident shock wave is characterized by the parameter $\alpha = \Delta p/(\rho a^2)$, where ρ is the density of the medium, a the velocity of sound in the medium, Δp the pressure difference across the front of the incident cylindrical shock wave S, and α will be assumed small.

We choose as initial parameters of the problem the velocity of propagation of the incident wave, which in the present cases coincides with the velocity of sound a in the medium, and the maximum diameter $2l$ of the cross section of the cylinder C. Let S^- denote the front of the reflected shock wave, $\alpha\Phi_0$ the velocity potential of the incident wave and let the diffraction begin at $t = 0$, where t is the time.

Under the assumption that the flow of the shock wave around the contour C is irrotational and isentropic, then a dimensionless formulation the problem of diffraction may be reduced to the determination of the perturbation velocity potential $\varphi(x,y,\tau)$, which satisfies with an accuracy up to quantities of the third order the equation [2]

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial\tau^2}=(k-1)\frac{\partial\varphi}{\partial\tau}\frac{\partial^2\varphi}{\partial\tau^2}+2\left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\,\partial\tau}+\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y\,\partial\tau}\right),\quad \tau=\frac{at}{l} \qquad (1.1)$$

and the conditions

$$\frac{\partial\varphi}{\partial n}=-\alpha\frac{\partial\Phi_0}{\partial n}+\varepsilon_\tau'(s,\tau) \quad \text{on } C$$
$$\varphi=0 \quad \text{for } \tau\leqslant 0, \qquad \text{on } S^- \text{ for } \tau>0 \qquad (1.2)$$

Here k is the adiabatic coefficient, (n,s) is the natural system of coordinates connected with the contour, $\varepsilon(s,\tau)$ is the magnitude of the deformation of the contour C under the influence of the incident wave

$$\varepsilon(s,\tau)=k_1 p(s,\tau) \quad \text{or} \quad \varepsilon(s,\tau)=\lambda_0\frac{\partial\varphi}{\partial\tau} \qquad (\lambda_0=-\rho a^2k_1) \qquad (1.3)$$

where p is the pressure on the contour created by the incident and reflected waves, and k_1 is the coefficient of rigidity of the contour, which is assumed to be known.

We will solve the system (1.1), (1.2) by setting

$$\varphi(x,y,\tau)=\varphi_1(x,y,\tau)+\varphi_2(x,y,\tau)+\ldots \qquad (1.4)$$

where φ_i is a quantity which is of the ith order in the small parameter α.

Substituting (1.4) into (1.1), (1.2), we obtain the following system of equations for the potentials φ_1 and φ_2 :

$$\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}-\frac{\partial^2\varphi_1}{\partial\tau^2}=0, \qquad \frac{\partial^2\varphi_2}{\partial x^2}+\frac{\partial^2\varphi_2}{\partial y^2}-\frac{\partial^2\varphi_2}{\partial\tau^2}=F(x,y,\tau)$$
$$\frac{\partial\varphi_1}{\partial n}=-\alpha\frac{\partial\Phi_0}{\partial n}+\lambda_0\frac{\partial^2\varphi_1}{\partial\tau^2}, \qquad \frac{\partial\varphi_2}{\partial n}=\lambda_0\frac{\partial^2\varphi_2}{\partial\tau^2} \quad \text{on } C \qquad (1.5)$$
$$\varphi_1=\varphi_2=0 \quad \text{for } \tau\leqslant 0, \qquad \text{on } S^- \text{ for } \tau>0$$

where F is the right-hand side of Equation (1.1) for $\varphi=\varphi_1(x,y,\tau)$.

The following theorem may be proved by the method of [1] :

Theorem 1. The two-dimensional problem described by the system (1.5) for the diffraction of a cylindrical weak shock wave at a cylinder of arbitrary cross section C is equivalent to two mixed Cauchy problems for the functions φ_1 and φ_2 in the space (x,y,τ) , also describable by the system (1.5), or is equivalent to the auxiliary external problem of supersonic flow of a gas at Mach number $M_1=\sqrt{2}$ and small angle of attack about a hollow cylinder corresponding to the contour C and semi-infinite along the τ-axis($\tau\geqslant 0$) .

The equation of the leading edge of the semi-infinite cylinder takes the form

$$\tau = A\ (x,\ y), \qquad y = f\ (x) \tag{1.6}$$

where $\tau = A$ is the equation of the incident wave front and $y = f(x)$ is the equation of the contour C.

By virtue of Theorem 1 we will solve the Cauchy problem (1.5) by Volterra's method [3]. Then for the determination of φ_1 and φ_2 on the surface of the hollow semi-infinite cylinder in the auxiliary problem we obtain the integro-differential equations

$$\begin{aligned}\varphi_1\ (x_0,\ y_0,\ \tau_0) &= \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iiint\limits_{\Sigma}\left[\varphi_1\ (x,\ y,\ \tau)\frac{\partial V}{\partial n} - V\ \frac{\partial\varphi_1}{\partial n}\right]ds\ d\tau\right\}\\ \varphi_2\ (x_0,\ y_0,\ \tau_0) &= \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iiint\limits_{\Sigma}\left[\varphi_2\ (x,\ y,\ \tau)\frac{\partial V}{\partial n} - \right.\right.\\ &\left.\left. - \left(\frac{\partial\varphi_2}{\partial n} - \lambda_0\frac{\partial\varphi_1}{\partial\tau}\frac{\partial^2\varphi_1}{\partial\tau^2}\right)V\right]ds\ d\tau + \iiint\limits_{T} FV dx\ dy\ d\tau\right\}\end{aligned} \tag{1.7}$$

where Σ is the portion of the surface of the cylinder which is cut out by the cone of influence from the point $(x_0,\ y_0,\ \tau_0)$, T is the volume bounded by the surface Σ, the surface of the cone of influence and the portion of the wave surface in the auxiliary problem which is cut out by the given cone of influence, and V is the Volterra function

$$V = \ln\frac{(\tau_0 - \tau) + \sqrt{(\tau_0 - \tau)^2 - (x_0 - x)^2 - (y_0 - y)^2}}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2}} \tag{1.8}$$

The potentials φ_1 and φ_2 at an arbitrary point of the perturbed region are expressed in quadratures through the values φ_1 and φ_2 on the surface of the cylinder.

In this manner the solution of the diffraction problem formulated above is reduced to the solution of Equation (1.7). In solving (1.7) numerically it is convenient to replace the singular function $\partial V/\partial n$ by the functions

$$K\ (\tau_0 - \tau,\ x_0 - x,\ y_0 - y) = \int\frac{\partial V}{\partial n}\,d\tau, \qquad K_1(\tau_0 - \tau, y_0 - y, x_0 - x) = \int K\ d\tau \tag{1.9}$$

which are continuous in the whole region Σ, including its boundary, and which vanishes at the point (x_0, y_0, τ_0) lying on the surface of the cylinder of the auxiliary problem. Equations (1.7) are then reduced to nonsingular equations for the potentials φ_1 and φ_2.

2. Solution of particular two-dimensional diffraction problems. 1. We shall consider the diffraction of a cylindrical weak shock wave at a semi-infinite thin plate $(y = 0,\ x \geqslant 0)$. We take the center of the cylindrical wave to be at the point $(x = m,\ y = -R)$ and the characteristic linear dimension to be equal to $l = a\cdot 1$ sec. Furthermore, we shall assume that $k_1 = 0$.

In view of Theorem 1 the given problem is equivalent to the problem of flow of a supersonic gas stream at $M_1 = \sqrt{2}$ and a small angle of atack around a semi-infinite plane wing (Fig. 1) bounded by the curves

$$\tau = -R + \sqrt{(x-m)^2 + R^2}, \quad y = 0, \qquad x \geqslant 0 \tag{2.1}$$

The auxiliary problem will be solved by Volterra's method.

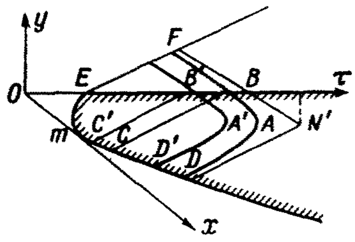

Fig. 1

The potentials φ_1 and φ_2 at an arbitrary point (x_0, y_0, τ_0) of the perturbed region, for which the region of integration Σ_1 does not extend beyond the wing, may be expressed immediately by Formulas

$$\begin{aligned}\varphi_1(x_0, y_0, \tau_0) &= \frac{\alpha}{\pi}\iint_{\Sigma_1} \frac{\partial \Phi_0}{\partial y}\frac{\partial V}{\partial \tau_0}\, dx\, d\tau \\ \varphi_2(x_0, y_0, \tau_0) &= \frac{1}{\pi}\iiint_T F(x, y, \tau)\frac{\partial V}{\partial \tau_0}\, dx\, dy\, d\tau\end{aligned} \tag{2.2}$$

In order to determine the potentials φ_1 and φ_2 at an arbitrary point of the perturbed region for which the region of integration extends beyond the wing, it is necessary to take into account the influence of the side edge of the wing. Thus we have

$$\begin{aligned}\varphi_1(x_0, y_0, \tau_0) &= \frac{\alpha}{\pi}\iint_{\Sigma_1} \frac{\partial \Phi_0}{\partial y}\frac{\partial V}{\partial \tau_0}\, dx\, d\tau + \frac{1}{\pi}\iint_{\sigma} Q_1 \frac{\partial V}{\partial \tau_0}\, dx\, d\tau \\ \varphi_2(x_0, y_0, \tau_0) &= \frac{1}{\pi}\iint_{\sigma} Q_2 \frac{\partial V}{\partial \tau_0}\, dx\, d\tau + \frac{1}{\pi}\iiint_T F(x, y, \tau)\frac{\partial V}{\partial \tau_0}\, dx\, dy\, d\tau\end{aligned} \tag{2.3}$$

Here Σ_1 denotes the area $ABEmDA$ and σ the area $BFEB$ in Fig. 1, while Q_1 and Q_2 denote the values of $\partial\varphi_1/\partial y$ and $\partial\varphi_2/\partial y$, respectively, on σ.

In order to determine Q_1 and Q_2 we write the expressions for φ_1 and φ_2 in the region σ. Since φ_1 and φ_2 are equal to zero on σ, then we obtain the following integral equations for the determination of Q_1 and Q_2:

$$\begin{aligned}&\iint_{\sigma_0} Q_1 \frac{\partial V}{\partial \tau_0}\, dx\, d\tau + \alpha \iint_{\Sigma_0} \frac{\partial \Phi_0}{\partial y}\frac{\partial V}{\partial \tau_0}\, dx\, d\tau = 0 \\ &\iint_{\sigma_0} Q_2 \frac{\partial V}{\partial \tau_0}\, dx\, d\tau + \iiint_{T_0} F(x, y, \tau)\frac{\partial V}{\partial \tau_0}\, dx\, dy\, d\tau = 0\end{aligned} \tag{2.4}$$

Here $\sigma_0 + \Sigma_0$ is the region of integration cut out by the cone of influence (Fig.1) from the point $(x_0, 0, \tau_0)$ lying in σ and T_0 is the corresponding volume.

We introduce the characteristic coordinates

$$\mu = \tau + x, \quad \nu = \tau - x$$

The transformation of Equations (2.4) can be carried out without difficulty [4], and finally we obtain

$$\varphi_1(x_0, y_0, \tau_0) = \frac{1}{\pi} \iint_{\Sigma_{10}} \frac{\partial \Phi_0}{\partial y} \frac{\partial V}{\partial \tau_0} \, dx \, d\tau$$
$$\varphi_2(x_0, y_0, \tau_0) = \frac{1}{\pi} \left\{ \iiint_T F(x, y, \tau) \frac{\partial V}{\partial \tau_0} \, dx \, dy \, d\tau - \iint_{\Sigma_1 - \Sigma_{10}} F \frac{\partial V}{\partial \tau_0} \, dx \, d\tau \right\} \tag{2.5}$$

where Σ_{10} is the area $ABCDA$ of the portion of the surface Σ_1 in Fig. 1.

For known values of φ_1 and φ_2 the pressure p at any point of the perturbed region may be determined, right up to the surface of the wing.

For $k_1 \neq 0$ but sufficiently small, the solution may be represented as a series in k_1, or it may be determined from the integro-differential equations.

2. We now shall consider a more general problem of the diffraction of a cylindrical weak shock wave at a plate of unit length moving along the x-axis with an arbitrary velocity U at zero angle of attack.

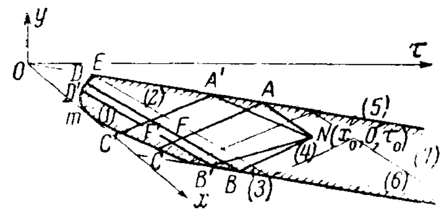

Fig. 2

By virtue of Theorem 1 we will solve the auxiliary problem of flow around a semi-infinite wing (Fig. 2) bounded by the curves

$$\tau = -R + \sqrt{(x - m)^2 + R^2} \tag{2.6}$$

$$x = M_0 \tau, \quad x = 1 + M_0 \tau, \quad M_0 = U / a$$

We devide the surface of the wing into subdomains (1), (2), (3), (4) and so forth, as shown in Fig. 2, and we restrict ourselves to the case $M_0 < 1$. The solution of the problem for the case $M_0 > 1$ is obtained even more easily, and the solution for a plane wave is described in [1].

As in the solution of the previous perticular problem, we find that:

a) at the points of the perturbed region which are influenced by the points on the surface of the wing lying in the subdomains (1) and (2), or (3), the potentials φ_1 and φ_2 are of the same form as solutions (2.2) and (2.5), respectively, and are obtained in a similar manner.

b) at those points of the perturbed region which are influenced by points on the surface of the wing lying in the subdomain (4), the values of φ_1 and φ_2 are equal to

$$\varphi_1(x_0, y_0, \tau_0) = \frac{\alpha}{\pi} \iint\limits_{\Sigma_0+EC} \frac{\partial \Phi_0}{\partial y} \frac{\partial V}{\partial \tau_0} \, dx \, d\tau \tag{2.7}$$

$$\varphi_2(x_0, y_0, \tau_0) = \frac{1}{\pi} \left\{ \iiint\limits_{T} F(x, y, \tau) \frac{\partial V}{\partial \tau_0} \, dx \, dy \, d\tau - \iint\limits_{\Sigma_1 - \Sigma_0} F(x, 0, \tau) \frac{\partial V}{\partial \tau_0} \, dx \, d\tau \right\} \tag{2.8}$$

where Σ_0 is the area $NAFDmCFBN$, the portion of the surface Σ_1 in Fig. 2.

It can be shown that the problem of diffraction of plane or cylindrical weak shock waves on a cascade of flat-plate profiles may be similarly solved by quadratures.

If instead of a plate we have a thin airfoil, then the problem of diffraction is also solvable by quadratures, but only for $U = 0$.

3. Formulation and solution of two-dimensional problems of diffraction of cylindrical elastic waves. In investigating the problems of diffraction of elastic waves we shall assume that the elastic medium obeys a nonlinear law of elasticity, in particular, the law of Murnaghan [5]

$$\begin{gathered} \sigma_{xx} = \lambda A_0 + 2\mu (1 + \beta_1 A_0) \varepsilon_{xx} + \beta_2 A_0^2 - \beta_1 A_1 + \beta_3 (\varepsilon_{xx}^2 + {}^1/_4 \varepsilon_{xy}^2 + {}^1/_4 \varepsilon_{xz}^2) \\ \sigma_{xy} = \mu (1 + \beta_1 A_0) \varepsilon_{xy} + \beta_3 [(\varepsilon_{xx} + \varepsilon_{yy}) \varepsilon_{xy} + {}^1/_2 \varepsilon_{xz} \varepsilon_{yz}] \end{gathered} \tag{3.1}$$

where λ, μ, β_1, β_2 and β_3 are elastic constants, and A_0, A_1 and A_2 are the invariants of the deformation tensor. The expressions for σ_{yy}, σ_{zz}, σ_{yz} and σ_{zx} are obtained from (3.1) by cyclic permutations. Moreover, we shall treat those elastic media for which $\beta_1 = \beta_3 = 0$, and assuming small deformations, we shall investigate the effect of the nonlinear terms in (3.1).

For simplicity we shall consider that the incident cylindrical elastic wave is describable by potentials $\alpha\Phi_0$ and Ψ_0 of longitudinal and transverse waves of the form

$$\Phi_0 = f(x, y, z, \tau), \quad \Psi_0 = 0, \quad \tau = a_0 t / l, \quad a_0^2 = (\lambda + 2\mu)/\rho \tag{3.2}$$

where a_0 is the velocity of propagation of longitudinal wave, ρ is the density of the medium, and l is the characteristic linear dimension of the problem.

Let the cylindrical elastic wave (3.2) impinge upon a contour C of arbitrary shape and let diffraction commence at $\tau = 0$ ($t = 0$).

We introduce the potentials Φ and Ψ of the longitudinal and transverse waves

$$u = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}, \qquad v = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x}, \qquad w = u + iv \tag{3.3}$$

where w is the vector of the displacement of an arbitrary point of the elastic medium.

Then in the absence of external forces the equations of motion of the elastic medium (3.1) may be put in terms of dimensionless variables in the form

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}-\frac{\partial^2\varphi}{\partial\tau^2}=-\left(\frac{\beta^2}{\rho a_0^2}\right)(\Delta\varphi)^2$$
$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{1}{b^2}\frac{\partial^2\psi}{\partial\tau^2}=0, \qquad b^2=\frac{\mu}{\rho a_0^2} \tag{3.4}$$

where b is the dimensionless velocity of propagation of transverse waves, and

$$\varphi=\Phi-\Phi_0, \quad \Psi=\Psi-\Psi_0 \tag{3.5}$$

We shall solve the problem for the following boundary and initial conditions:

$$q_1(s,\tau)=\varepsilon(s,\tau), \qquad q_2(s,\tau)=f_1(s,\tau) \qquad \text{on } C \tag{3.6}$$

where n, s are the natural coordinates associated with the contour, q_1, q_2 are the components of the displacement vector along the normal and tangent to the contour, respectively, and $\varepsilon(s,\tau)$ is the magnitude of the deformation of the contour C under the action of the elastic wave. For simplicity we shall assume that

$$\varepsilon(s,\tau)={}^1/_2\, k_2(\sigma_{xx}+\sigma_{yy}), \qquad f_1=k_3\sigma_{sn} \tag{3.7}$$

and S^- is the front of the reflected wave, and k_2 and k_3 are constants.

We shall solve the problems (3.4), (3.6) and (3.7) by setting

$$\varphi(x,y,\tau)=\varphi_1(x,y,\tau)+\varphi_2(x,y,\tau)+\ldots$$
$$\psi(x,y,\tau)=\psi_1(x,y,\tau) \tag{3.8}$$

The system of equations for φ_1, φ_2, ψ_1 becomes

$$\Delta\varphi_1-\frac{\partial^2\varphi_1}{\partial\tau^2}=0, \qquad \Delta\varphi_2-\frac{\partial^2\varphi_2}{\partial\tau^2}=F(x,y,\tau), \qquad \Delta\psi_1-\frac{1}{b^2}\frac{\partial^2\psi_1}{\partial\tau^2}=0$$
$$\frac{\partial\varphi_1}{\partial n}=-\alpha\frac{\partial\Phi_0}{\partial n}+\frac{\partial\psi_1}{\partial s}+\lambda_1\frac{\partial^2\varphi_1}{\partial\tau^2}, \qquad \frac{\partial\varphi_2}{\partial n}=\lambda_1\frac{\partial^2\varphi_2}{\partial\tau^2} \quad \text{on } C$$
$$\frac{\partial\psi_1}{\partial n}-\alpha\frac{\partial\Phi_0}{\partial s}-\frac{\partial\dot\varphi_1}{\partial s}-\frac{\lambda_2}{b^2}\frac{\partial^2\psi_1}{\partial\tau^2}+2\lambda_1\lambda_2\frac{\partial^3\varphi_1}{\partial\tau\,\partial s^2} \quad \text{on } C \tag{3.9}$$
$$\lambda_1=k_2(\lambda+\mu), \qquad \lambda_2=\frac{\mu k_3}{1-\mu k_3}$$
$$\varphi_1=\varphi_2=\psi_1=0 \quad \text{for } \tau\leqslant 0, \qquad \text{на } S^- \quad \text{for } \tau>0$$

where $F(x,y,\tau)$ denotes the right-hand side of Equation (3.4) for $\varphi=\varphi_1$.

As in the solution of the diffraction of plane weak shock waves, we have the following Theorem [2]:

Theorem 2. The diffraction of a cylindrical elastic wave on the contour C, described by the system (3.9), is equivalent to the external problem of two supersonic flows of an ideal gas, at $M_1 = \sqrt{2}$ and $M_2 = \sqrt{1+b^2}$, over a hollow cylinder Q corresponding to the contour C and semi-infinite in extent along the τ-axis $(\tau \geqslant 0)$, or it is equivalent to three mixed Cauchy problems, which are also described by the system (3.9) in the space (x, y, τ).

The equation of the leading edge of the cylinder in the auxiliary problem is given by Equation (1.6).

By virtue of Theorem 2 we can make use of Volterra's method to determine φ_1, φ_2 and ψ_1. We then obtain for φ_1, φ_2 and ψ_1 on the surface of the cylinder Q a system of two-dimensional integro-differential equations of the form

$$\varphi_1(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial \tau_0}\left\{\iint\limits_{\Sigma_1}\left[\varphi_1 \frac{\partial V}{\partial n} - V\frac{\partial \varphi_1}{\partial n}\right] ds\, d\tau\right.$$

$$\varphi_2(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial \tau_0}\left\{\iint\limits_{\Sigma_1}\left[\varphi_2 \frac{\partial V}{\partial n} - \left(\frac{\partial \varphi_2}{\partial n} + \lambda_1 \frac{\partial \varphi_1}{\partial \tau}\frac{\partial^2 \varphi_1}{\partial \tau^2} - \right.\right.\right. \tag{3.10}$$

$$\left.\left.\left. - \lambda_1 \frac{\partial \varphi_1}{\partial \tau}\frac{\partial^2 \varphi_1}{\partial s\, \partial \tau}\right) V\right] ds\, d\tau + \iiint\limits_{T} F(x, y, \tau)\, V\, dx\, dy\, d\tau\right\}$$

$$\psi_1(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial \tau_0}\left\{\iint\limits_{\Sigma_2}\left[\psi_1 \frac{\partial V_1}{\partial n} - V_1\frac{\partial \psi_1}{\partial n}\right] ds\, d\tau\right\}$$

where Σ_1 and Σ_2 are the portions of the surface of the cylinder cut out by the cones of influence

$$(\tau_0 - \tau)^2 - (x_0 - x)^2 - (y_0 - y)^2 = 0$$

$$b^2(\tau_0 - \tau)^2 - (x_0 - x)^2 - (y_0 - y)^2 = 0$$

from the point (x_0, y_0, τ_0) lying on the surface of the cylinder, V is the Volterra function (1.8), and

$$V_1(\tau_0 - \tau, x_0 - x, y_0 - y) = V\,[b(\tau_0 - \tau), x_0 - x, y_0 - y] \tag{3.11}$$

Thus the general diffraction problem reduces to the solution of the system (3.10), which for a given contour C may be solved numerically, or which for certain contours C can be obtained by quadratures.

We shall consider particular problems of diffraction of elastic cylindrical waves.

1. Let us consider the problem of diffraction of a cylindrical elastic wave on a semi-infinite cut $(y = 0,\ x \geqslant 0)$, which is rigidly attached to the surrounding medium. Let the center of the wave be at the point $(x = m, y = -R)$.

In accordance with Theorem 2, instead of the diffraction problem we shall solve two auxiliary mixed problems (Fig.1), in which the half-plane Q is bounded by the curves (2.1).

Then following Volterra's method for the solution of the three-dimensional Cauchy problem for the potentials φ_1 and ψ_1 at an arbitrary point $(x_0, 0, \tau_0)$, for which the region of integration does not extend beyond the boundaries of the half-plane Q, we obtain Expressions

$$\begin{aligned}\varphi_1(x_0, 0, \tau_0) &= -\frac{1}{\pi}\iint_{\Sigma_1}\frac{\partial\psi_1}{\partial x}\frac{\partial V}{\partial\tau_0}\,dx\,d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_1}\frac{\partial\Phi_0}{\partial y}\frac{\partial V}{\partial\tau_0}\,dx\,d\tau \\ \psi_1(x_0, 0, \tau_0) &= \frac{1}{\pi}\iint_{\Sigma_2}\frac{\partial\varphi_1}{\partial x}\frac{\partial V_1}{\partial\tau_0}\,dx\,d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_2}\frac{\partial\Phi_0}{\partial x}\frac{\partial V_1}{\partial\tau_0}\,dx\,d\tau\end{aligned} \tag{3.12}$$

Similarly, for the point $(x_0, 0, \tau_0)$ for which the region of integration exceeds the limits of the half-plane

$$\begin{aligned}\varphi_1(x_0, 0, \tau_0) &= -\frac{1}{\pi}\iint_{\Sigma_{10}}\frac{\partial\psi_1}{\partial x}\frac{\partial V}{\partial\tau_0}\,dx\,d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_{10}}\frac{\partial\Phi_0}{\partial y}\frac{\partial V}{\partial\tau_0}\,dx\,d\tau \\ \psi_1(x_0, 0, \tau_0) &= \frac{1}{\pi}\iint_{\Sigma_{20}}\frac{\partial\varphi_1}{\partial x}\frac{\partial V_1}{\partial\tau_0}\,dx\,d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_{20}}\frac{\partial\Phi_0}{\partial x}\frac{\partial V_1}{\partial\tau_0}\,dx\,d\tau\end{aligned} \tag{3.13}$$

where Σ_{10} and Σ_{20} are portions of the surfaces Σ_1 and Σ_2, or $A'B'C'D'A'$, as shown in Fig. 1.

Let us consider Equations (3.12). Introducing the characteristic coordinates $\mu = \tau + x$, $\nu = \tau - x$ in the first of equations (3.12) and the corresponding coordinates in the second equation, we can show that the derivative may be taken outside the double integral. Then eliminating ψ_1 and φ_1, we can put Equations (3.12) in the form

$$\begin{aligned}\varphi_1(x_0, 0, \tau_0) &= -\frac{\partial}{\pi^2\partial x_0}\iint_{\Sigma_1}\frac{\partial V}{\partial\tau_0}\,dx\,d\tau\,\frac{\partial}{\partial x}\iint_{\Sigma_2}\varphi_1\frac{\partial V_1}{\partial\tau}\,dx'd\tau' + F_1(x_0, \tau_0) \\ \psi_1(x_0, 0, \tau_0) &= -\frac{1}{\pi^2}\frac{\partial}{\partial x_0}\iint_{\Sigma_2}\frac{\partial V_1}{\partial\tau_0}\,dx\,d\tau\,\frac{\partial}{\partial x}\iint_{\Sigma_1}\psi_1\frac{\partial V}{\partial\tau}\,dx'd\tau' + F_2(x_0, \tau_0)\end{aligned} \tag{3.14}$$

where F_1 and F_2 are known quadratures in Equations (3.12).

It can be shown that

$$\begin{aligned}&\left|\frac{1}{\pi^2}\frac{\partial}{\partial x_0}\iint_{\Sigma_1}\frac{\partial V}{\partial\tau_0}\,dxd\tau\,\frac{\partial}{\partial x}\iint_{\Sigma_2}\frac{\partial V_1}{\partial\tau}\,dx'd\tau'\right| < 1, \qquad 0 \leqslant b < 1 \\ &\left|\frac{1}{\pi^2}\frac{\partial}{\partial x_0}\iint_{\Sigma_2}\frac{\partial V_1}{\partial\tau_0}\,dxd\tau\,\frac{\partial}{\partial x}\iint_{\Sigma_1}\frac{\partial V}{\partial\tau}\,dx'd\tau'\right| < 1, \qquad 0 \leqslant b < 1\end{aligned} \tag{3.15}$$

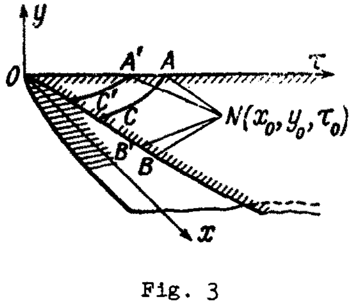

Fig. 3

and consequently Equations (3.14) are solvable for φ_1 and ψ_1, and an approximate solution may be obtained by iteration. The first iteration provides approximate expressions for φ_1 and ψ_1 of acceptable accuracy for practical purposes. Successive iterations contribute not more than 2 – 3% over a wide range of variations of the parameter b.

The system (3.13) is solved in a similar manner,

The determination of the potential φ_2 does not differ in principle from the determination of φ_2 in the problems solved in Section 2, Subsection 1, and it has the form (2.2) or (2.3).

The problem for $\Psi_0 \neq 0$ is solved in an analogous fashion.

2. Let a longitudinal elastic cylindrical wave impinge on a plate of unit length $(0 \leqslant x \leqslant 1)$, which moves with velocity U along the x-axis.

By virtue of Theorem 2 we will solve the auxiliary Cauchy problem.

We divide the surface of the semi-infinite plate into regions (1), (2), (3), ..., (Fig. 2). The potentials φ_1 and ψ_1 at points $(x_0, 0, \tau_0)$ of regions (1) and (2), or (3), have the forms (3.12) and (3.13). At points of region (4), however

$$\begin{aligned}\varphi_1(x_0, 0, \tau_0) &= -\frac{1}{\pi}\iint_{\Sigma_0} \frac{\partial\psi_1}{\partial x}\frac{\partial V}{\partial\tau_0}\,dx\,d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_0}\frac{\partial\Phi_0}{\partial y}\frac{\partial V}{\partial\tau_0}\,dx\,d\tau \\ \psi_1(x_0, 0, \tau_0) &= \frac{1}{\pi}\iint_{\Sigma_0'}\frac{\partial\varphi_1}{\partial x}\frac{\partial V_1}{\partial\tau_0}\,dx\,d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_0'}\frac{\partial\Phi_0}{\partial x}\frac{\partial V_1}{\partial\tau_0}\,dx\,d\tau\end{aligned} \tag{3.16}$$

where Σ_0' denotes the area $NA'F'D'C'F'B'N$ in Fig. 2.

Equations (3.16) are solved for φ_1 and ψ_1 just as were Equations (3.12) or (3.13). In the case $U = U(\tau)$ the solution is constructed analogously. The solution for $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ (but sufficiently small) may be obtained in the form of series in λ_1 and λ_2.

The solution of the problem of diffraction of elastic waves on cascaded profiles consisting of rectilinear segments or on n slits is solved similarly.

3. We shall consider the problem of diffraction of a circular-cylindrical elastic wave on a wedge of half-angle $0 \leqslant \beta \leqslant \pi/2$. Let the center of the cylindrical wave lie on the axis of symmetry of the wedge and outside the wedge.

By virtue of Theorem 2 we shall solve the auxiliary mixed Cauchy problem for the potentials φ_1 and ψ_1, or the problem of flow around the corresponding three-dimensional semi-infinite $(\tau \geqslant 0)$ corner with the plane of symmetry $y = 0$ (Fig.3).

Solving the given problem described by the equations and boundary and initial conditions (3.9) by Volterra's method and noting that

$$\varphi_1(x, y, \tau) = +\varphi_1(x, -y, \tau), \qquad \psi_1(x, y, \tau) = -\psi_1(x, -y, \tau) \tag{3.17}$$

we obtain the following equations for φ_1 and ψ_1 :

$$\begin{aligned}\varphi_1(x_0, y_0, \tau_0) &= -\frac{1}{\pi}\iint_{\Sigma_1+\Sigma_{10}'}\frac{\partial\psi_1}{\partial q_1}\frac{\partial V}{\partial\tau_0}\,dq_1 d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_1+\Sigma_{10}'}\frac{\partial\Phi_0}{\partial q_2}\frac{\partial V}{\partial\tau_0}\,dq_1 d\tau \\ \psi_1(x_0, y_0, \tau_0) &= \frac{1}{\pi}\iint_{\Sigma_2-\Sigma_{20}'}\frac{\partial\varphi_1}{\partial q_1}\frac{\partial V_1}{\partial\tau_0}\,dq_1 d\tau + \frac{\alpha}{\pi}\iint_{\Sigma_2-\Sigma_{20}'}\frac{\partial\Phi_0}{\partial q_1}\frac{\partial V_1}{\partial\tau_0}\,dq_1 d\tau\end{aligned} \tag{3.18}$$

where

$$q_1 = x\cos\beta + y\sin\beta, \qquad q_2 = -x\sin\beta + y\cos\beta$$

and Σ_{10}', Σ_{20}' are portions of the surfaces Σ_1 and Σ_2 ($NACBN$ and $NA'C'B'N$ in Fig. 3), bounded by the curves

$$\tau = \tau_0 - \sqrt{(q_{10} - q_1)^2\cos^2\beta + (q_{10} + q_1)^2\sin^2\beta}, \quad \tau = A(q_1), \quad q_2 = 0, \quad y > 0 \quad (AC)$$

$$\tau = \tau_0 - b^{-1}\sqrt{(q_{10} - q_1)^2\cos^2\beta + (q_{10} + q_1)^2\sin^2\beta}, \quad \tau = A(q_1), \quad q_2 = 0, \quad y > 0 \quad (A'C) \tag{3.19}$$

respectively.

Eliminating φ_1 and ψ_1 from the integrals of Equations (3.18), we obtain

$$\varphi_1(x_0, y_0, \tau_0) = -\frac{1}{\pi^2}\frac{1}{\partial q_{10}} \iint\limits_{\Sigma_1+\Sigma_{10}'} \frac{\partial V}{\partial \tau_0}\, dq_1 d\tau \frac{\partial}{\partial q_1} \iint\limits_{\Sigma_2-\Sigma_{20}'} \varphi_1 \frac{\partial V_1}{\partial \tau}\, dq_1' d\tau' + F_3(x_0, y_0, \tau_0)$$

$$\psi_1(x_0, y_0, \tau_0) = -\frac{1}{\pi^2}\frac{\partial}{\partial q_{10}} \iint\limits_{\Sigma_2-\Sigma_{20}'} \frac{\partial V_1}{\partial \tau_0}\, dq_1 d\tau \frac{\partial}{\partial q_1} \iint\limits_{\Sigma_1+\Sigma_{10}'} \psi_1 \frac{\partial V}{\partial \tau}\, dq_1' d\tau' + F_4(x_0, y_0, \tau_0) \qquad (3.20)$$

where F_3 and F_4 are known integrals.

Equations (3.20) for φ_1 and ψ_1 may be solved by method of iteration, as in the preceding problems.

For $b = 0$ the integrals in Equations (3.20) vanish and $\psi_1 = 0$. The expression for φ_1 gives the solution of the problem of diffraction of a weak shock wave or acoustic wave on the wedge.

N o t e . The author takes this opportunity to note a correction to the paper [2], where in the solution of the problem of diffraction of a weak shock wave on a prolate spheroid, the value $\xi = 1$ was taken for the surface of the spheroid. Actually the surface of the spheroid should be described by Equation $\xi = a$, $a \geqslant 1$, where a is an arbitrary constant. For $\xi = 1$ the prolate spheroid degenerates into a section of the z-axis.

In conclusion, the author thanks E.A. Ivanov for a number of valuable remarks he made in reviewing this paper.

BIBLIOGRAPHY

1. Filippov, I.G., K teorii difraktsii slabykh udarnykh voln okolo konturov proizvol'noi formy (On the theory of diffraction of weak shock waves at contours of arbitrary shape). *PMM* Vol.27, № 1, 1963.

2. Filippov, I.G., K teorii lineinykh prostranstvennykh nestatsionarnykh zadach difraktsii i nekotorye nelineinye zadachi (On the theory of linear three-dimensional unsteady problems of diffraction and certain nonlinear problems). *PMM* Vo.27, № 4, 1963.

3. Goursat, E., Kurs matematicheskogo analiza (A Course in Mathematical Analysis). Gostekhteoretizdat, Vol.3, Part 1, 1933.

4. Krasil'shchikova, E.A., Krylo konechnogo razmakha v szhimaemom potoke (The wing of finite span in a compressible flow). M.-L., 1952.

5. Murnaghan, F., Finite deformation of an elastic solid. Amer. J. Math., Vol.59, pp.235-260, 1937.

Translated by F.A.L.